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# Burch's inequality and the depth of the blow up rings of an ideal

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**Abstract**

Let $(A,\mathfrak{m})$ be a local noetherian ring with infinite residue field and $I$ an ideal of $A$. Consider $R_A(I)$ and $G_A(I)$, respectively, the Rees algebra and the associated graded ring of $I$, and denote by $l(I)$ the analytic spread of $I$. Burch's inequality says that $l(I)+\inf\{\operatorname{depth} A/I^n,\ n\geq 1\}\leq \dim(A)$, and it is well known that equality holds if $G_A(I)$ is Cohen–Macaulay. Thus, in that case one can compute the depth of the associated graded ring of $I$ as $\operatorname{depth} G_A(I)=l(I)+\inf\{\operatorname{depth} A/I^n,\ n\geq 1\}$. We study when such an equality is also valid when $G_A(I)$ is not necessarily Cohen–Macaulay, and we obtain positive results for ideals with analytic deviation less or equal than one and reduction number at most two. In those cases we may also give the value of $\operatorname{depth} R_A(I)$. 

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## 1. Introduction

Let $(A,\mathfrak{m})$ be a local noetherian ring and $I\subseteq A$ an ideal of $A$. Consider $R_A(I)=\bigoplus_{n\geq 0} I^nt^n\subseteq A[t]$, $G_A(I)=\bigoplus_{n\geq 0} I^n/I^{n+1}$, respectively, the Rees algebra and the associated graded ring of $I$ (usually called the blow up rings of $I$). The Cohen–Macaulay property of these rings has been extensively studied from several points of view by

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many authors, and deep results have been obtained in recent years by giving fine conditions on the ideal $I$ for the Cohen–Macaulayness of its blow up rings, see for example [31] or [15] and the literature cited there for detailed information about.

On the contrary, much less is known about their depths when the blow up rings fail to be Cohen–Macaulay, and precise formulae have been obtained only in few cases. For simplicity, assume from now on that the residue field $A/\mathfrak{m}$ is infinite. Recall that $J \subseteq I$ is said to be a *reduction* of $I$ if $I^{n+1} = JI^n$ for some $n \geq 0$, and *minimal reduction* if no other reduction of $I$ is contained in $J$. The *analytic spread* $l(I)$ of $I$ is then defined as the minimal number of generators of a (any) minimal reduction of $I$, and the *analytic deviation* of $I$ as the difference $\mathrm{ad}(I) = l(I) - \mathrm{ht}(I)$ $(\geq 0)$. Ideals with analytic deviation zero are usually called equimultiple, and $\mathfrak{m}$-primary ideals are always equimultiple. Furthermore, for a given reduction $J$ of $I$ the *reduction number of $I$ with respect to $J$* is defined as $r_J(I) = \min\{n \mid I^{n+1} = JI^n\}$, and the *reduction number* of $I$ as $r(I) = \min\{r_J(I), J \text{ a minimal reduction of } I\}$. Finally, if $h := \mathrm{ht}(I)$ we define the local reduction number of $I$ as $r_h(I) = \max\{r(I_\mathfrak{p}),\ \mathfrak{p} \in V(I) \text{ with } \mathrm{ht}(\mathfrak{p}) = h\}$.

In [5] Brodmann proved that if $A$ is Cohen–Macaulay and $I \subseteq A$ is a generically complete intersection ideal of $A$ with $\mathrm{ad}(I) = 1$ and $r(I) = 0$, then $\operatorname{depth} G_A(I) = \min\{\dim A, \operatorname{depth} A/I + \mathrm{ht}(I) + 1\}$ and $\operatorname{depth} R_A(I) = \min\{\dim A + 1, \operatorname{depth} A/I + \mathrm{ht}(I) + 2\}$, see also [17]. The same formulae have been shown to hold if $I$ is generically a complete intersection and $\mathrm{ad}(I) = 1$ with $r(I) \leq 1$, see [32].

On the other hand, Burch's inequality [6] says that for any ideal $I$, $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} + l(I) \leq \dim A$, and it is known [11] that equality holds if $\mathrm{ht}(I) > 0$ and $G_A(I)$ is Cohen–Macaulay, see also [27]. Hence, if $G_A(I)$ is Cohen–Macaulay one can write $\operatorname{depth} G_A(I) = \inf\{\operatorname{depth} A/I^n,\ n \geq 1\} + l(I)$. It is clear that we cannot expect such a formula in general, since there are local rings $(A, \mathfrak{m})$ which are Cohen–Macaulay with $\operatorname{depth} G_A(\mathfrak{m}) = 0$, see [12] for many different examples. But as a consequence of the results we prove in this paper one may formulate the following:

**Theorem 1.1.** *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay ring and $I \subseteq A$ an ideal of $A$ such that $\mathrm{ad}(I) \leq 1$, $r(I) \leq 2$ and $r_h(I) \leq 1$, where $h = \mathrm{ht}(I)$. Assume in addition that $I$ is unmixed if either $\mathrm{ad}(I) = 1$ or $r(I) = 2$. Then*

$$\begin{aligned} &\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} + l(I) - 1 \\ &\quad \leq \operatorname{depth} G_A(I) \\ &\quad \leq \inf\{\operatorname{depth} A/I^n,\ n \geq 1\} + l(I) + 1. \end{aligned}$$

*Furthermore, if either $r(I) \leq 1$ and $I$ is generically a complete intersection when $\mathrm{ad}(I) = 1$, or $\operatorname{depth} A/I \neq \operatorname{depth} A/I^2$ and $\mathrm{ht}(I) > 0$ then*

$$\operatorname{depth} G_A(I) = \inf\{\operatorname{depth} A/I^n,\ n \geq 1\} + l(I).$$

To simplify, we shall call the value $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\}$ the *Burch number* of $I$ and denote it by $B(I)$. Note that Brodmann proves in [4] that the depths of $A/I^n$

have a stable value, which he computes in [5] for almost complete intersection ideals. In this case, this asymptotic value coincides with $B(I)$ but this does not occur in general, see Section 3. In fact, in this paper we shall compute for the ideals under the hypothesis of Theorem 1.1 their Burch number in Section 3, while in Sections 4 and 5 we compute the depths of their associated graded rings. Then, by direct comparison of both computations we may directly formulate the statement in Theorem 1.1. Also, by using the results proved by Huckaba and Marley [20] which relate the depths of $G_A(I)$ and $R_A(I)$ we may give the precise value of depth $R_A(I)$ for most of the cases we study.

In order to prove our results we usually reduce to the $\mathrm{ht}(I)=0$ case. This is possible since under the hypothesis of Theorem 1.1 it can be shown that there exist minimal reductions of $I$ with particularly nice properties, see [1] or [15] for similar results including some of the special cases we treat. We stablish these technical results as well as some other ones that we shall use along the paper in Section 2. Finally, in Section 6 we compute the depths of $G_A(I^n)$ and $R_A(I^n)$, $n \geq 1$, for the family of ideals we consider in this work. Under some extra hypothesis we prove that, in fact, depth $G_A(I^n) =$ depth $G_A(I)$ and depth $R_A(I^n) =$ depth $R_A(I)$ for all $n \geq 1$. This might be seen as a natural extension of the well known fact, first proved by Valla [30] for complete intersection ideals, that for any ideal $I$ if $R_A(I)$ (resp. $G_A(I)$) is Cohen–Macaulay then $R_A(I^n)$ (resp. $G_A(I^n)$) is Cohen–Macaulay for all $n \geq 1$, see [16, Corollary 47.6] (resp. [18, Corollary 4.6]).

## 2. Preliminaries

In what follows $(A, \mathfrak{m})$ will denote a noetherian local ring of dimension $d$. Let $I$ be an ideal of $A$. Recall that $I$ is said to be *generically a complete intersection* if $\mu(I_{\mathfrak{p}}) = \mathrm{ht}(I)$ for all minimal prime ideals $\mathfrak{p} \in \mathrm{Min}(A/I)$, and *unmixed* if $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$ for all associated prime ideals $\mathfrak{p} \in \mathrm{Ass}(A/I)$. Often we will put $l$ for the analytic spread $l(I)$. If $S$ is a noetherian graded ring over $A$, $M$ its maximal homogeneous ideal and $N$ a graded $S$-module, we shall denote by $H^i_M(N)$ the $i$th graded local cohomology module of $N$ with respect to $M$, and by $a_i(N) = \sup\{n \mid [H^i_M(N)]_n \neq 0\} (< \infty)$. Recall that $a(N) := a_{\dim N}(N)$ is usually called the *a-invariant* of $N$. We will denote by $\mathscr{M}$ the maximal homogeneous ideal of $R_A(I)$ and put $G := G_A(I)$ and $R := R_A(I)$. We will use the following result frequently throughout the paper.

**Lemma 2.1** (Goto and Huckaba [13, Lemma 2.2]). *Let $S = \bigoplus_{n \geq 0} S_n$ be a noetherian graded ring with $(S_0, \mathfrak{m}_0)$ a local ring. Let $N$ be a finitely generated graded $S$-module with $N_n = 0$ for all $n \gg 0$. Then for any integers $i, n$ we have an isomorphism $H^i_M(N)_n \simeq H^i_{\mathfrak{m}_0}(N_n)$ of $S_0$-modules, where $M$ is the maximal homogeneous ideal of $S$.*

For an element $a \in I^n \setminus I^{n+1}$ we denote its initial form of degree $n$ by $a^* \in I^n/I^{n+1} \hookrightarrow G_A(I)$. Valabrega and Valla [29] gave a criterion to determine when a sequence of initial forms of elements of $A$ provides a regular sequence in the associated graded ring. In particular, for elements of degree one their result states:

**Lemma 2.2** (Valabrega and Valla [29, Corollary 2.7]). *Let $(A,\mathfrak{m})$ be a local ring and $I$ an ideal of $A$. Let $a_1,\dots,a_k$ be a sequence of elements in $I\setminus I^2$. Then, $a_1^*,\dots,a_k^*$ form a regular sequence in $G_A(I)$ if and only if $a_1,\dots,a_k$ form a regular sequence in $A$ and $I^{n+1}\cap(a_1,\dots,a_k)=I^n(a_1,\dots,a_k)$ for all $n\geq 0$. In this case we have that $G_A(I)/(a_1^*,\dots,a_k^*)\simeq G_{A/(a_1,\dots,a_k)}(I/(a_1,\dots,a_k))$.*

The following results will be used to reduce several proofs to the case of height zero.

**Remark 2.3.** Assume that $r_h(I)\leq 1$ and let $\mathfrak{p}\supseteq I$ be a prime ideal with $\mathrm{ht}(\mathfrak{p})=h$. Then $I_\mathfrak{p}$ is a $\mathfrak{p}A_\mathfrak{p}$-primary ideal with $r(I_\mathfrak{p})\leq 1$ and it is well known, see for example [22, Proposition 4.25], that in this case the reduction number $r_J(I_\mathfrak{p})$ is independent of the minimal reduction $J$. In particular, $I_\mathfrak{p}^2=J'I_\mathfrak{p}$ for all reductions $J'$ of $I_\mathfrak{p}$. On the other hand, note that if $I$ is unmixed then $r_h(I)=\max\{r(I_\mathfrak{p})\mid \mathfrak{p}\in\mathrm{Min}\,A/I\}$.

For an ideal $K$, we denote by $\bar{K}$ its integral closure.

**Lemma 2.4.** *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal of $A$. Let $J=(a_1,\dots,a_h)$ be a minimal reduction of $I$ such that $I^3=JI^2$. Assume that either $I$ is integrally closed or $I$ is unmixed with $r_h(I)\leq 1$. Then $a_1^*,\dots,a_h^*$ is a regular sequence in $G_A(I)$.*

**Proof.** Since $I$ is equimultiple $a_1,\dots,a_h$ is a regular sequence in $A$ so by Lemma 2.2 it suffices to verify the equality $I^2\cap J=JI$. If $I$ is integrally closed then, by [21, Theorem 3.1], $\overline{I^2}\cap J=JI$ and the claim is clear. Assume now that $I$ is unmixed with $r_h(I)\leq 1$. Then it suffices to show that $I_\mathfrak{p}^2\cap J_\mathfrak{p}\subseteq J_\mathfrak{p}I_\mathfrak{p}$ for all $\mathfrak{p}\in\mathrm{Ass}\,A/JI$. We may proceed as in [19, Section 2] to show that $\mathrm{Ass}\,A/JI\subseteq\mathrm{Ass}\,A/J\cup\mathrm{Ass}\,A/I$. Let $\mathfrak{p}\in\mathrm{Ass}\,A/JI$. If $\mathfrak{p}\in\mathrm{Ass}\,A/I=\mathrm{Min}\,A/I$ then $I_\mathfrak{p}^2=J_\mathfrak{p}I_\mathfrak{p}$ since $J_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$ and $r_h(I)\leq 1$ by assumption. If $\mathfrak{p}\in\mathrm{Ass}\,A/J\setminus\mathrm{Ass}\,A/I$ we have that $I_\mathfrak{p}=A_\mathfrak{p}$ since $I$ is unmixed. In any case $I_\mathfrak{p}^2\cap J_\mathfrak{p}\subseteq J_\mathfrak{p}I_\mathfrak{p}$ as we wanted. □

**Lemma 2.5.** *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an ideal with* $\mathrm{ad}(I)=1$. *Assume that $I$ is generically a complete intersection. Let $J$ be a minimal reduction of $I$. Then, there exists a minimal system of generators $a_1,\dots,a_h$, $a$ of $J$ satisfying:*

(a) *$a_1,\dots,a_h$ is an $A$-regular sequence.*
(b) *$I_\mathfrak{p}=(a_1,\dots,a_h)_\mathfrak{p}$ for all $\mathfrak{p}\in\mathrm{Min}(A/I)$.*
(c) *$((a_1,\dots,a_h)^m : a^n)\cap I^m=(a_1,\dots,a_h)^m$ for all $n,m\geq 1$.*
(d) *If $h\geq 1$ and $I^2=JI$, $(a_1,\dots,a_h)^i\cap I^n=(a_1,\dots,a_h)I^{n-i}$ for all $n\geq 1$, $i=1,\dots,n-1$. In particular, $a_1^*,\dots,a_h^*$ is a regular sequence in $G_A(I)$.*

**Proof.** Conditions (a) and (b) follow from [32, Lemma 2.2] and for (c) and (d) it suffices to apply [19, Remark 2.1(iii) and Lemma 2.5]. □

For the case of analytic deviation one ideals with reduction number at most 2 we first need the following definition (see [1,15]).

**Definition 2.6.** Let $I$ be an ideal and $J$ a minimal reduction of $I$. Then a system of generators $a_1,\dots,a_l$ for $J$ is said to be *good* if
(a) $(a_1,\dots,a_i)_{\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$ for all $\mathfrak{p}\in V(I)$ with $i=\mathrm{ht}(\mathfrak{p})\le l$.
(b) $a_i\notin\mathfrak{p}$ if $\mathfrak{p}\in\operatorname{Ass}A/(a_1,\dots,a_{i-1})\setminus V(I)$ for any $1\le i\le l$.

By [1, Lemma 2.3] or [15, Lemma 2.1] one has that any minimal reduction $J$ of $I$ has a good system of generators.

**Lemma 2.7.** *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an ideal with* $\mathrm{ad}(I)=1$. *Assume that $I$ is unmixed and $r_h(I)\le 1$. Let $J$ be a minimal reduction of $I$. Then, there exists a minimal system of generators $a_1,\dots,a_h$, $a$ of $J$ satisfying*:
(a) *$a_1,\dots,a_h$ is an $A$-regular sequence.*
(b) *$I_{\mathfrak{p}}^2=(a_1,\dots,a_h)_{\mathfrak{p}}I_{\mathfrak{p}}$ for all $\mathfrak{p}\in\operatorname{Min}(A/I)$.*
(c) *$((a_1,\dots,a_h):a)\cap I^2=(a_1,\dots,a_h)I$.*
(d) *If $h\ge 1$ and $I^3=JI^2$, $I^{n+1}\cap(a_1,\dots,a_h)=(a_1,\dots,a_h)I^n$ for all $n\ge 1$. In particular, $a_1^*,\dots,a_h^*$ is a regular sequence in $G_A(I)$.*

**Proof.** Let $a_1,\dots,a_h$, $a$ be a *good* system of generators for $J$. Then (a) follows from [1, Lemma 2.6] or [15, Lemma 2.2]. Taking into account Remark 2.3, (b) is immediate since for any $\mathfrak{p}\in\operatorname{Min}A/I$ one has that $(a_1,\dots,a_h)_{\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$.

We proof now (c). If $h=0$ we want to show that $(0:a)\cap I^2=0$, or equivalently that $(0:a)_{\mathfrak{p}}\cap I_{\mathfrak{p}}^2=0$ for all $\mathfrak{p}\in\operatorname{Min}A$. If $\mathfrak{p}\not\supseteq I$ then, by condition (b) of Definition 2.6, we have that $a\notin\mathfrak{p}$ and so $(0:a)_{\mathfrak{p}}=0$. If $\mathfrak{p}\supseteq I$ then $I_{\mathfrak{p}}^2=0$. In any case $(0:a)_{\mathfrak{p}}\cap I_{\mathfrak{p}}^2=0$ as we wanted. Assume now that $h>0$. We want to see that $((a_1,\dots,a_h):a)_{\mathfrak{p}}\cap I_{\mathfrak{p}}^2=(a_1,\dots,a_h)_{\mathfrak{p}}I_{\mathfrak{p}}$ for all $\mathfrak{p}\in\operatorname{Ass}A/(a_1,\dots,a_h)I$. Let $\mathfrak{p}\in\operatorname{Ass}A/(a_1,\dots,a_h)I\subseteq\operatorname{Ass}A/(a_1,\dots,a_h)\cup\operatorname{Ass}A/I$. If $\mathfrak{p}\in\operatorname{Ass}A/I=\operatorname{Min}A/I$ then $I_{\mathfrak{p}}^2=(a_1,\dots,a_h)_{\mathfrak{p}}I_{\mathfrak{p}}$. Assume $\mathfrak{p}\in\operatorname{Ass}A/(a_1,\dots,a_h)$. Then $\mathrm{ht}(\mathfrak{p})=h$. So, if $\mathfrak{p}\supseteq I$ then $\mathfrak{p}\in\operatorname{Min}A/I$. On the contrary we have that $\mathfrak{p}\not\supseteq I$, $a\notin\mathfrak{p}$ and so $((a_1,\dots,a_h):a)_{\mathfrak{p}}=(a_1,\dots,a_h)_{\mathfrak{p}}$. In any case, $((a_1,\dots,a_h):a)_{\mathfrak{p}}\cap I_{\mathfrak{p}}^2=(a_1,\dots,a_h)_{\mathfrak{p}}I_{\mathfrak{p}}$ and (c) is proved.

For (d) assume first that $n=1$. $I^2\cap(a_1,\dots,a_h)\subseteq((a_1,\dots,a_h):a)\cap I^2=(a_1,\dots,a_h)I$ by (c) and so $I^2\cap(a_1,\dots,a_h)=(a_1,\dots,a_h)I$. Let $n>1$ and $x\in I^{n+1}\cap(a_1,\dots,a_h)=JI^n\cap(a_1,\dots,a_h)$ since $r_J(I)\le 2$. Then, $x=b_1a_1+\cdots+b_ha_h+ba$ with $b_i,b\in I^n$. Thus, $b\in I^n\cap(a_1,\dots,a_h)=(a_1,\dots,a_h)I^{n-1}$ by induction. Therefore, $ba\in(a_1,\dots,a_h)I^n$ and $x\in(a_1,\dots,a_h)I^n$. Moreover, by Lemma 2.2 one has that $a_1^*,\dots,a_h^*$ is regular in $G_A(I)$. □

There are many examples of equimultiple ideals with reduction number 1. In particular, recent work by Corso et al. [10], Corso and Polini [8,9] and Polini and Ulrich [24] show how to get such ideals by linkage.

Next one is an example of equimultiple ideal with reduction number 2 which is integrally closed.

**Example 2.8.** Let $A=k[[X,Y,T_1,\dots,T_n]]/(X^3Y)=k[[x,y,t_1,\dots,t_n]]$, where $k$ is a field and $n\geq 3$. $A$ is a $(n+1)$-dimensional Cohen–Macaulay ring and $I=(xy,t_1,\dots,t_n)\subseteq A$ is equimultiple with $\mathrm{ht}(I)=n$ and reduction number 2, which is integrally closed since $A/I$ is reduced.

Our next example shows an analytic deviation one ideal with reduction number two which is unmixed and has local reduction number less or equal than one.

**Example 2.9.** Let $A=K[[x,y,z,w]]$, where $k$ is a field, and let $\mathfrak{p}\subseteq A$ be the ideal generated by the definition ideal of the projective monomial curve given by $x=u^8$, $y=u^5v^3$, $z=u^3v^5$, $w=v^8$. Morales and Simis prove [23] that a minimal set of generators for $\mathfrak{p}$ is given by the maximal minors $\Delta_{12},\Delta_{13},\Delta_{14},\Delta_{24}$ of the matrix

$$\begin{pmatrix} x & z & y^4 & wy^3 \\ y & w & xz^3 & z^4 \end{pmatrix}$$

and that $\mathfrak{p}$ is a prime ideal with $\mathrm{ht}(\mathfrak{p})=2$, $l(\mathfrak{p})=3$ and $r(\mathfrak{p})=1$ since $\mathfrak{p}^2=(\Delta_{12},\Delta_{13},\Delta_{24})\mathfrak{p}$.

Set $J=\mathfrak{p}^2$. Then $J$ is unmixed since $\mathfrak{p}^2=\mathfrak{p}^{(2)}$ (see [23, Proposition 2.3]), $\mathrm{ht}(J)=2$, $l(J)=3$ and $r(J)=2$. In fact, $(\Delta_{12}^2,\Delta_{13}^2,\Delta_{24}^2)$ is a minimal reduction of $J$. Furthermore, $r_2(J)=1$ since $A_{\mathfrak{p}}$ is a regular ring of dimension two.

In addition, to obtain analytic deviation one ideals with reduction number less or equal than two and local reduction number one we may also proceed in the following way.

**Example 2.10.** Let $A$ be a Cohen–Macaulay ring and $I\subset A$ a generically complete intersection ideal with $\mathrm{ht}(I)=h$. Assume that $I$ is unmixed, $r(I)=0$ and $\mathrm{ad}(I)=1$. Then, $I=(a_1,\dots,a_h,a_{h+1})$ where $a_1,\dots,a_h$ is a regular sequence and $I_{\mathfrak{p}}=(a_1,\dots,a_h)_{\mathfrak{p}}$ for all $\mathfrak{p}\supseteq I$ with $\mathrm{ht}(\mathfrak{p})=h$. Set $B=A/(a_1^2)$ and $J=(\bar{a}_1,\dots,\bar{a}_h,\bar{a}_{h+1})\subset B$. Since $B/J=A/I$, $J$ is unmixed and $\mathrm{ht}(J)=h-1$. Furthermore, for any $\mathfrak{q}\supseteq I$ with $\mathrm{ht}(\mathfrak{q})=h-1$ we have that $J_{\mathfrak{q}}=(\bar{a}_1,\dots,\bar{a}_h)_{\mathfrak{q}}$ and $J_{\mathfrak{q}}^2=(\bar{a}_2,\dots,\bar{a}_h)_{\mathfrak{q}}J_{\mathfrak{q}}$. So $J_{\mathfrak{q}}$ is equimultiple with reduction number 1, that is, the local reduction number of $J$ equals 1. Since $J^2=(\bar{a}_2,\dots,\bar{a}_h,\bar{a}_{h+1})J$ we also have that $\mathrm{ad}(J)=1$ and $r(J)=1$.

For instance, let $A=k[a,b,c,d,e]$ ($k$ a field) and $I=(f_1,f_2,f_3)$ where

$$f_1=5abcde-a^5-b^5-c^5-d^5-e^5,$$

$$f_2=ab^3c+bc^3d+a^3be+cd^3e+ade^3,$$

$$f_3=a^2bc^2+b^2cd^2+a^2d^2e+ab^2e^2+c^2de^2$$

(see [19, Example 4.7]). This is an almost complete intersection ideal such that $\mathrm{ht}(I)=2$. Then, for $B=A/(f_1^2)$ and $J=I/(f_1^2)$ we have that $J$ is an unmixed ideal with $\mathrm{ht}(J)=1$,

$\mathrm{ad}(I)=1$ and $r(J)=1$. Moreover, one may verify by using CoCoA [7] that $\operatorname{depth} A/I=0$ and so $\operatorname{depth} B/J = 0$, whereas $\dim B/J = 3$.

Similarly, by considering the almost complete intersection ideals obtained in [26, Example] and proceeding in the way above described, one may get ideals $J$ such that $\operatorname{depth} B/J < \dim B/J - 1$.

**Lemma 2.11.** *Let $(A,\mathfrak{m})$ be a local ring and $I$ an ideal of $A$. Assume that $a \in I$ satisfies $(a)\cap I^{n+1} = aI^n$ for all $n \geq 0$. Let $A_1 = A/(a)$ and $I_1 = I/(a) \subseteq A_1$. If $a$ is part of a minimal system of generators of a minimal reduction of $I$, then $l(I_1) = l(I) - 1$.*

**Proof.** Straightforward. □

We summarize in the following lemma some known results which link the depths of the associated graded ring and the Rees ring of an ideal. We shall use them to determine the depth of the Rees algebra.

**Lemma 2.12.** *Let $(A,\mathfrak{m})$ be a local ring, and $I$ an ideal.*

(a) [20, Theorem 3.10] $\operatorname{depth} G_A(I) \leq \operatorname{depth} A$, *and if* $\operatorname{depth} G_A(I) < \operatorname{depth} A$ *then* $\operatorname{depth} R_A(I) = \operatorname{depth} G_A(I) + 1$.
(b) [27, Theorem 7.1] *Assume that $A$ is Cohen–Macaulay and* $\mathrm{ht}(I) > 0$. *Then $R_A(I)$ is a Cohen–Macaulay ring if and only if $G_A(I)$ is a Cohen–Macaulay ring with $a(G_A(I)) < 0$.*
(c) [3, Theorem 5.6] *or* [28, Theorem 1.4] *Assume that $G_A(I)$ is Cohen–Macaulay and put $\mathscr{A}(I) = \{\mathfrak{p} \in V(I) \mid l(I_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})\}$. Then,*
$$a(G_A(I)) = \max\{\max\{r(I_\mathfrak{p}) - l(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathscr{A}(I), \mathrm{ht}(\mathfrak{p}) < l(I)\}, r(I) - \mathrm{ht}(I)\}.$$

## 3. Computing the Burch number

We begin this section by proving a lemma that will be very useful to control depths in proceeding by induction.

**Lemma 3.1.** *Let $(A,\mathfrak{m})$ be a local ring and $I$ an ideal of $A$. Let $a_1,\dots,a_k$ be a family of elements in $I \setminus I^2$ such that $a_1^*,\dots,a_k^*$ is a regular sequence in $G_A(I)$. Let $A_k = A/(a_1,\dots,a_k)$ and $I_k = I/(a_1,\dots,a_k)$, $k \geq 1$.*

(a) *If* $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$ *then* $\operatorname{depth} A_k/I_k^2 = \operatorname{depth} A/I^2$.
(b) *If* $\operatorname{depth} A/I < \operatorname{depth} A/I^2$ *then* $\operatorname{depth} A_k/I_k^2 = \operatorname{depth} A/I - 1$.
(c) *If* $\operatorname{depth} A/I = \operatorname{depth} A/I^2$ *then* $\operatorname{depth} A_k/I_k^2 \geq \operatorname{depth} A/I - 1$.

**Proof.** Since the family $a_1^*,\dots,a_k^*$ is regular in $G_A(I)$, we have that $a_1,\dots,a_k$ form a regular sequence in $A$ and that $(a_1,\dots,a_i)\cap I^2 = (a_1,\dots,a_i)I$ for all $i=1,\dots,k$.

For all $i=0,\dots,k-1$ we can consider the morphism $A_i/I_i^2 \to A_{i+1}/I_{i+1}^2$ (where $A_0 := A$ and $I_0 := I$), which kernel $(a_{i+1})/(a_{i+1})\cap I_i^2 = (a_{i+1})/(a_{i+1})I_i \simeq A_i/I_i$ is isomorphic to

$A/I$. Thus, for $i = 0, \ldots, k-1$ we have exact sequences

$$0 \to A/I \to A_i/I_i^2 \to A_{i+1}/I_{i+1}^2 \to 0. \tag{1}$$

Assume that $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$. Then, applying the *depth-lemma* to the exact sequences (1) successively for $i = 0, \ldots, k-1$, we get that $\operatorname{depth} A_i/I_i^2 = \operatorname{depth} A/I^2$ for all $i = 1, \ldots, k$ and (a) is proved.

Assume that $\operatorname{depth} A/I < \operatorname{depth} A/I^2$. Then, from (1) for $i = 0$, we obtain that $\operatorname{depth} A_1/I_1^2 = \operatorname{depth} A/I - 1$. In particular, $\operatorname{depth} A_1/I_1^2 < \operatorname{depth} A/I$ and by (a) we have now that $\operatorname{depth} A_i/I_i^2 = \operatorname{depth} A_1/I_1^2 = \operatorname{depth} A/I - 1$ for all $i = 1, \ldots, k$. This proves (b).

Assume that $\operatorname{depth} A/I = \operatorname{depth} A/I^2$. In this case, by taking $i = 0$ in (1) we have

$$\operatorname{depth} A/I = \operatorname{depth} A/I^2 = \operatorname{depth} A_1/I_1^2$$

or

$$\operatorname{depth} A/I^2 = \operatorname{depth} A/I = \operatorname{depth} A_1/I_1^2 + 1$$

or

$$\operatorname{depth} A_1/I_1^2 > \operatorname{depth} A/I = \operatorname{depth} A/I^2.$$

If $\operatorname{depth} A_1/I_1^2 = \operatorname{depth} A/I - 1$, from (a) it follows that $\operatorname{depth} A_1/I_1^2 = \operatorname{depth} A_k/I_k^2 = \operatorname{depth} A/I - 1$. If $\operatorname{depth} A_1/I_1^2 > \operatorname{depth} A/I$, we have by (b) that $\operatorname{depth} A_2/I_2^2 = \cdots = \operatorname{depth} A_k/I_k^2 = \operatorname{depth} A/I - 1$. Otherwise $\operatorname{depth} A_1/I_1^2 = \operatorname{depth} A/I$ and repeating the above argument if necessary for $i=1, \ldots, k-1$ we may conclude that $\operatorname{depth} A_i/I_i^2 \geq \operatorname{depth} A/I - 1$ for all $i = 2, \ldots, k$. In any case we have $\operatorname{depth} A_i/I_i^2 \geq \operatorname{depth} A/I - 1$ for all $i = 1, \ldots, k$ and (c) is proved. □

For equimultiple ideals with reduction number 1 we may compute the Burch number in the following way:

**Proposition 3.2.** *Let $(A, \mathfrak{m})$ be a local Cohen–Macaulay ring and $I$ an equimultiple ideal of $A$ with $r(I) \leq 1$. Then,* $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} = \operatorname{depth} A/I$.

**Proof.** The proof is by induction on $h := \operatorname{ht}(I)$. Let $J = (a_1, \ldots, a_h)$ be a minimal reduction of $I$ with $I^2 = JI$. Then, the sequence $a_1^*, \ldots, a_h^*$ is regular in $G_A(I)$. If $h = 0$, for all $n \geq 2$ one has $I^n = 0$ and so $\operatorname{depth} A/I^n = \operatorname{depth} A = \dim A \geq \operatorname{depth} A/I$.

Suppose that $h > 0$ and let $A_1 = A/(a_1)$, $I_1 = I/(a_1)$. Then, $A_1$ is a Cohen–Macaulay ring, and $I_1$ is an equimultiple ideal with $r(I_1) \leq 1$ and $\operatorname{ht}(I_1) = h - 1$. On the other hand, for all $n \geq 2$ we can consider the epimorphism $A/I^n \to A_1/I_1^n$, which kernel $(a_1)/I^n \cap (a_1) \simeq (a_1)/(a_1)I^{n-1}$ is isomorphic to $A/I^{n-1}$. So, for each $n \geq 2$ we have the exact sequence

$$0 \to A/I^{n-1} \to A/I^n \to A_1/I_1^n \to 0 \tag{2}$$

and hence, $\operatorname{depth} A/I^n \geq \min\{\operatorname{depth} A/I^{n-1}, \operatorname{depth} A_1/I_1^n\} \geq \operatorname{depth} A/I$, by induction on $n$ and $h$. □

For reduction number 2 we have:

**Proposition 3.3.** *Let* $(A,\mathfrak{m})$ *be a local Cohen–Macaulay ring and* $I$ *an equimultiple ideal of* $A$ *with* $r(I)\leq 2$. *Assume that* $I$ *is integrally closed or* $I$ *is unmixed with* $r_h(I)\leq 1$. *Then*

$$\inf\{\operatorname{depth} A/I^n,\ n\geq 1\}\geq \min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I-1\}.$$

*Furthermore*,
(a) *equality holds if* $h>0$ *and* $\operatorname{depth} A/I\neq \operatorname{depth} A/I^2$, *and*
(b) $\inf\{\operatorname{depth} A/I^n,\ n\geq 1\}=\min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\}$ *if* $h=0$.

**Proof.** First of all note that (b) is obvious since if $h=0$, $I^n=0$ for all $n\geq 3$.

We will prove now that $\inf\{\operatorname{depth} A/I^n, n\geq 1\}\geq \min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I-1\}$. For $h=0$ the inequality holds trivially from (b). Assume $h>0$ and let $J=(a_1,\dots,a_h)$ be a minimal reduction of $I$ with $I^3=JI^2$. Then, by Lemma 2.4 the family $a_1^*,\dots,a_h^*$ is a regular sequence in $G_A(I)$. Then $A_1=A/(a_1)$ is a Cohen–Macaulay ring and $I_1=I/(a_1)$ is an equimultiple ideal with $r(I_1)\leq 2$ and $\operatorname{ht}(I_1)=h-1$. Using the same argumentation as is Proposition 3.2 we get for all $n\geq 3$ exact sequences as in (2)

$$0\to A/I^{n-1}\to A/I^n\to A_1/I_1^n\to 0.$$

Therefore,

$$\begin{aligned}\operatorname{depth} A/I^n &\geq \min\{\operatorname{depth} A/I^{n-1}, \operatorname{depth} A_1/I_1^n\}\\ &\geq \min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I-1, \operatorname{depth} A_1/I_1^2\}\end{aligned}$$

by induction on $n$ and $h$ and taking into account that $A_1/I_1\simeq A/I$. Since, by Lemma 3.1, $\operatorname{depth} A_1/I_1^2\geq \operatorname{depth} A/I-1$ we have the assertion.

For (a) suppose that $\operatorname{depth} A/I\neq \operatorname{depth} A/I^2$. If $\operatorname{depth} A/I^2<\operatorname{depth} A/I$ then $\min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I-1\}=\operatorname{depth} A/I^2$ and hence $\inf\{\operatorname{depth} A/I^n,\ n\geq 1\}=\operatorname{depth} A/I^2$. Assume that $\operatorname{depth} A/I<\operatorname{depth} A/I^2$. Then, from the exact sequences

$$0\to A_i/I_i^2\to A_i/I_i^3\to A_{i+1}/I_{i+1}^3\to 0 \tag{3}$$

for $i=0,\dots,h-1$, together with the fact that in this case $\operatorname{depth} A_i/I_i^2=\operatorname{depth} A/I-1$ for $i=0,\dots,h-1$ whereas $\operatorname{depth} A_h/I_h^3=\operatorname{depth} A_h=\operatorname{depth} A/I$, we obtain that $\operatorname{depth} A/I^3=\operatorname{depth} A/I-1$ and so $\inf\{\operatorname{depth} A/I^n,\ n\geq 1\}=\operatorname{depth} A/I-1$. □

**Lemma 3.4.** *Let* $(A,\mathfrak{m})$ *be a local Cohen–Macaulay ring and* $I$ *an ideal of* $A$ *of height* 0. *Assume that there exists* $a\notin Z_A(I)$. *Then*
(a) $\operatorname{depth} A/aI=\operatorname{depth} A/I$ *if* $\operatorname{depth} A/I<\dim A/I$.
(b) $\operatorname{depth} A/aI=\operatorname{depth} A/I-1$ *if* $\operatorname{depth} A/I=\dim A/I$.

**Proof.** Consider the exact sequences

$$0\to I\to A\to A/I\to 0, \tag{4}$$

$$0\to I/aI\to A/aI\to A/I\to 0. \tag{5}$$

Assume first that $\operatorname{depth} A/I < \dim A/I$. Then, from (4) we have that $\operatorname{depth} I = \operatorname{depth} A/I + 1$ and so $\operatorname{depth} I/aI = \operatorname{depth} A/I$, since $a \notin Z_A(I)$. Together with (5) this proves that $\operatorname{depth} A/aI = \operatorname{depth} A/I$ and so (a).

Assume now that $\operatorname{depth} A/I = \dim A/I = \dim A$. From (4) we obtain $\operatorname{depth} I = \dim A$, thus $\operatorname{depth} I/aI = \dim A - 1$. From (5) one has that $\operatorname{depth} A/aI = \dim A - 1 = \operatorname{depth} A/I - 1$ and (b) is proved. □

For analytic deviation 1 ideals with reduction number less or equal than 1 we have the following result that recovers the computations made by Brodmann [5] for almost complete intersection ideals:

**Proposition 3.5.** *Let* $(A, \mathfrak{m})$ *be a local Cohen–Macaulay ring and* $I$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$ *and* $r(I) \leq 1$. *Assume that* $I$ *is generically a complete intersection. Then*,

$$\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} = \min\{\operatorname{depth} A/I, \dim A/I - 1\}.$$

**Proof.** We will show that if $\operatorname{depth} A/I < \dim A/I$, then $\operatorname{depth} A/I^n = \operatorname{depth} A/I$ for $n \geq 2$ and if $\operatorname{depth} A/I = \dim A/I$, then $\operatorname{depth} A/I^n = \operatorname{depth} A/I - 1$ for $n \geq 2$. We will prove this by induction on $h$.

Suppose that $h = 0$. Then $I^{n+1} = aI^n$ for all $n \geq 1$, for certain $a \notin Z_A(I)$. If depth $A/I < \dim A/I$, we may apply Lemma 3.4 inductively on $n$ for all $n \geq 1$ to deduce that $\operatorname{depth} A/I^{n+1} = \operatorname{depth} A/I^n = \operatorname{depth} A/I$. Assume now that $\operatorname{depth} A/I = \dim A/I$. Then, by Lemma 3.4, $\operatorname{depth} A/I^2 = \operatorname{depth} A/aI = \operatorname{depth} A/I - 1$ and $\operatorname{depth} A/I^{n+1} = \operatorname{depth} A/aI^n = \operatorname{depth} A/I^n = \operatorname{depth} A/I - 1$ for all $n \geq 2$.

Now suppose that $h > 0$. Let $J = (a_1, \ldots, a_h, a_{h+1})$ be a minimal reduction of $I$ as in Lemma 2.5 with $I^2 = JI$. Then, the sequence $a_1, \ldots, a_h$ is regular in $G_A(I)$ and if we put $A_1 = A/(a_1)$ and $I_1 = I/(a_1)$, we have that $A_1$ is a Cohen–Macaulay ring and that $I_1$ is generically a complete intersection ideal with $\operatorname{ad}(I_1) = 1$, $r(I_1) \leq 1$ and $\operatorname{ht}(I_1) = h - 1$. Once more, from the exact sequences of (2) for all $n \geq 2$

$$0 \to A/I^{n-1} \to A/I^n \to A_1/I_1^n \to 0,$$

we obtain the assertion of the proposition by induction on $h$ and $n$. □

For reduction number less or equal than 2 and local reduction number less or equal than 1 we obtain:

**Proposition 3.6.** *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring and* $I$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$, $r(I) \leq 2$ *and* $r_h(I) \leq 1$, *where* $h = \operatorname{ht}(I)$. *Assume that* $I$ *is unmixed. Then*,

$$\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} \geq \min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I - 1\}.$$

*Furthermore*,

(a) *equality holds if* $\mathrm{ht}(I) > 0$ *and* $\operatorname{depth} A/I^2 \neq \operatorname{depth} A/I$, *and*

(b) $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} = \min\{\dim A/I - 1, \operatorname{depth} A/I\}$ *if* $h = 0$ *and* $\operatorname{depth} A/I = \operatorname{depth} A/I^2$.

**Proof.** Assume first that $h=0$. Then, there exists $a \notin Z(I^2)$ with $I^{n+1}=aI^n$ for all $n \geq 2$. If $\operatorname{depth} A/I^2 = \operatorname{depth} A/I$, then applying Lemma 3.4 successively to $I^n$ for $n \geq 2$ we obtain that $\operatorname{depth} A/I^n = \operatorname{depth} A/I$ if $\operatorname{depth} A/I < \dim A/I$, and $\operatorname{depth} A/I^n = \operatorname{depth} A/I - 1$ otherwise, for $n \geq 3$. Thus, in any case, $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} = \min\{\dim A/I - 1, \operatorname{depth} A/I\}$ and (b) is proved.

We will prove now the inequality of the statement. We first consider the case $h=0$. From (b) we may assume that $\operatorname{depth} A/I^2 \neq \operatorname{depth} A/I$. If $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$, from Lemma 3.4 we get that $\operatorname{depth} A/I^n = \operatorname{depth} A/I^2$, for $n \geq 3$. Finally, if $\operatorname{depth} A/I < \operatorname{depth} A/I^2$ one can deduce, applying again Lemma 3.4, that $\operatorname{depth} A/I^n \geq \operatorname{depth} A/I$ for all $n \geq 3$. Now assume that $h > 0$ and let $J = (a_1, \ldots, a_h, a_{h+1})$ be a minimal reduction of $I$ as in Lemma 2.7 with $a_1^*, \ldots, a_h^*$ regular in $G_A(I)$. From the exact sequences (2) for $n \geq 3$ we obtain that

$$\begin{aligned}\operatorname{depth} A/I^n &\geq \min\{\operatorname{depth} A/I^{n-1}, \operatorname{depth} A_1/I_1^n\}\\ &\geq \min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I - 1, \operatorname{depth} A_1/I_1^2\}\\ &\geq \min\{\operatorname{depth} A/I^2, \operatorname{depth} A/I - 1\},\end{aligned}$$

by induction on $n$ and $h$, and Lemma 3.1.

For (a) assume first that $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$. Then, from the inequality it follows that $\operatorname{depth} A/I^n \geq \operatorname{depth} A/I^2$ and so that $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} = \operatorname{depth} A/I^2$. On the other hand, if $\operatorname{depth} A/I < \operatorname{depth} A/I^2$ then, by Lemma 3.1, we have that $\operatorname{depth} A_h/I_h^2 = \operatorname{depth} A/I - 1$ and applying the results obtained for $h=0$ it follows that $\operatorname{depth} A_h/I_h^3 = \operatorname{depth} A/I - 1$. Using the exact sequences (3) for $i=0,\ldots,h-1$ as in Proposition 3.3 we get $\operatorname{depth} A/I^3 = \operatorname{depth} A/I - 1$ and so, in this case $\inf\{\operatorname{depth} A/I^n,\ n \geq 1\} = \operatorname{depth} A/I - 1$. $\square$

## 4. Equimultiple ideals

We begin by computing the depths of the form and Rees rings of equimultiple ideals with reduction number 1. In particular, for the Cohen–Macaulay property we recover [27, Proposition 7.4].

**Theorem 4.1.** *Let* $(A, \mathfrak{m})$ *be a local Cohen–Macaulay ring and* $I \subseteq A$ *an equimultiple ideal with* $r(I) \leq 1$. *Then*,

(a) $\operatorname{depth} G_A(I) = \operatorname{depth} A/I + \mathrm{ht}(I)$.

(b) *If* $\mathrm{ht}(I) \geq 2$, $\operatorname{depth} R_A(I) = \operatorname{depth} A/I + \mathrm{ht}(I) + 1$.

**Proof.** The proof is by reduction to the case $\mathrm{ht}(I) = 0$. Put $h = \mathrm{ht}(I)$. Assume that $h = 0$. Then $I^2 = 0$ and so $G_A(I) = A/I \oplus I$. Thus, $\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} I\} = \operatorname{depth}_A A/I$.

Assume now that $h > 0$ and let $J = (a_1, \ldots, a_h) \subseteq I$ be a minimal reduction of $I$ such that $I^2 = JI$. Then $a_1^*, \ldots, a_h^*$ is a regular sequence and $G_A(I)/(a_1^*, \ldots, a_h^*) \simeq G_{A_h}(I_h)$, where $A_h = A/(a_1, \ldots, a_h)$ and $I_h = I/(a_1, \ldots, a_h)$. Hence, $\operatorname{depth} G_A(I) = \operatorname{depth} G_{A_h}(I_h) + h$. Since $I_h$ is an equimultiple ideal with $\mathrm{ht}(I_h) = 0$ and $r(I_h) \leq 1$ we have that $\operatorname{depth} G_{A_h}(I_h) = \operatorname{depth} A_h/I_h = \operatorname{depth} A/I$ and consequently $\operatorname{depth} G_A(I) = \operatorname{depth} A/I + \mathrm{ht}(I)$ as wanted. For (b) it suffices to apply Lemma 2.12. □

**Corollary 4.2.** *Let $(A, \mathfrak{m})$ be a local Cohen–Macaulay ring and $I \subseteq A$ an equimultiple ideal with $r(I) \leq 1$. Then*, $\operatorname{depth} G_A(I) = \mathrm{B}(I) + l(I)$.

**Proof.** It suffices to apply Theorem 4.1 and Proposition 3.2. □

**Remark 4.3.** Let $(A, \mathfrak{m})$ be a local Cohen–Macaulay ring and $I \subseteq A$ an equimultiple ideal with $r(I) = 1$. Assume that $\mathrm{ht}(I) = 1$. Then, $R_A(I)$ is never Cohen–Macaulay. If so, then $G_A(I)$ would be Cohen–Macaulay with $a(G_A(I)) = r(I) - h(I) = 0$, which is a contradiction.

**Remark 4.4.** Let $(A, \mathfrak{m})$ be a local Cohen–Macaulay ring and $I \subseteq A$ an equimultiple ideal with $r(I) = 1$. Assume that $\mathrm{ht}(I) = 0$. Then, $I^n = 0$ for all $n > 1$ and $\operatorname{depth} R_A(I) = \min\{\operatorname{depth} A/I + 1, \dim A\}$. Furthermore the following are trivially equivalent:
(a) $R_A(I)$ is Cohen–Macaulay.
(b) $\operatorname{depth} G_A(I) \geq \dim A - 1$.
(c) $\operatorname{depth} A/I \geq \dim A - 1$.

Next, we consider the case of equimutilple ideals with reduction number 2.

**Theorem 4.5.** *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal of $A$. Assume that $r(I) \leq 2$ and either $I$ is integrally closed or $I$ is unmixed with $r_h(I) \leq 1$, where $h = \mathrm{ht}(I)$. Then*:

$$\begin{aligned}
&\min\{\operatorname{depth} A/I - 1, \operatorname{depth} A/I^2\} + \mathrm{ht}(I) \\
&\quad \leq \operatorname{depth} G_A(I) \\
&\quad \leq \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \mathrm{ht}(I).
\end{aligned}$$

*Furthermore*,
(a) *if* $\mathrm{ht}(I) = 0$, *then* $\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\}$, *and*
(b) *if* $\mathrm{ht}(I) > 0$ *and* $\operatorname{depth} A/I^2 \neq \operatorname{depth} A/I$, *then*

$$\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I - 1, \operatorname{depth} A/I^2\} + \mathrm{ht}(I).$$

**Proof.** Assume that $h = 0$. Then $I^3 = 0$. So $G_A(I) = A/I \oplus I/I^2 \oplus I^2$ and $\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} I/I^2, \operatorname{depth} I^2\}$. Applying the depth-lemma to the exact sequences

$$0 \to I \to A \to A/I \to 0,$$

$$0 \to I^2 \to A \to A/I^2 \to 0,$$

$$0 \to I/I^2 \to A/I^2 \to A/I \to 0,$$

$$0 \to I^2 \to I \to I/I^2 \to 0,$$

one has that $\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\}$.

Assume now that $h > 0$. Let $J = (a_1, \dots, a_h)$ be a minimal reduction of $I$ as in Lemma 2.4 and put $A_h = A/J$ and $I_h = I/J$. Since $a_1^*, \dots, a_h^*$ is a regular sequence in $G_A(I)$ we have $G_A(I)/(a_1^*, \dots, a_h^*) \simeq G_{A_h}(I_h)$ and so $\operatorname{depth} G_A(I) = \operatorname{depth} G_{A_h}(I_h) + h$. Moreover, $A_h$ is a Cohen–Macaulay ring and $I_h$ is an equimultiple ideal with $r(I_h) \leq 2$ and $\operatorname{ht}(I_h) = 0$. Therefore, applying the result obtained for height zero ideals we have that $\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} + h$. Finally, by Lemma 3.1, we have the following: If $\operatorname{depth} A/I \neq \operatorname{depth} A/I$ then $\min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} = \min\{\operatorname{depth} A/I - 1, \operatorname{depth} A/I^2\}$. If $\operatorname{depth} A/I = \operatorname{depth} A/I^2$, then $\min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} \geq \operatorname{depth} A/I - 1$. Thus, (b) is proved and we have, in any case, that the inequalities of the theorem are satisfied. $\square$

**Corollary 4.6.** *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal of $A$. Assume that $r(I) \leq 2$ and either $I$ is integrally closed or $I$ is unmixed and $r_h(I) \leq 1$, with $h = \operatorname{ht}(I)$. Then*

$$\mathrm{B}(I) + l(I) - 1 \leq \operatorname{depth} G_A(I) \leq \mathrm{B}(I) + l(I) + 1.$$

*Furthermore*, $\operatorname{depth} G_A(I) = \mathrm{B}(I) + l(I)$ *if either* $\operatorname{ht}(I) = 0$ *or* $\operatorname{ht}(I) > 0$ *and* $\operatorname{depth} A/I \neq \operatorname{depth} A/I^2$.

**Proof.** It follows directly from Theorem 4.5 and Proposition 3.3. $\square$

By again applying Lemma 2.12 we get for the Rees algebra:

**Theorem 4.7.** *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal of $A$. Assume that $r(I) \leq 2$ and either $I$ is integrally closed or $I$ is unmixed and $r_h(I) \leq 1$, with $h = \operatorname{ht}(I) \geq 3$. Then*:

$$\begin{aligned} &\min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \operatorname{ht}(I) \\ &\quad \leq \operatorname{depth} R_A(I) \\ &\quad \leq \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \operatorname{ht}(I) + 1. \end{aligned}$$

*Furthermore*, *if* $\operatorname{depth} A/I^2 \neq \operatorname{depth} A/I$, *then*

$$\operatorname{depth} R_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2 + 1\} + \operatorname{ht}(I).$$

**Example 4.8.** Consider the ideal $I=(xy,t_1,\dots,t_n)$ given in Example 2.8. It is easy to see that $\operatorname{depth} A/I=\dim A/I=1$ and $\operatorname{depth} A/I^2=0$. So in this case we have that $B(I)=\operatorname{depth} A/I^2=0$ and $\operatorname{depth} G_A(I)=B(I)+l(I)=n$ ($<\dim G_A(I)=n+1$). Furthermore, $\operatorname{depth} R_A(I)=n+1$ ($<\dim R_A(I)=n+2$).

## 5. Analytic deviation one ideals

First of all we recall the following formula for the case of analytic deviation one ideals with reduction number at most 1.

**Proposition 5.1** (Zarzuela [32, Theorem 3.1]). *Let $(A,\mathfrak{m})$ be a local Cohen–Macaulay ring and $I\subseteq A$ an ideal with $\operatorname{ad}(I)=1$ and $r(I)\leq 1$. Assume that $I$ is generically a complete intersection. Then*
(a) $\operatorname{depth} G_A(I)=\min\{\dim A, \operatorname{depth} A/I+\operatorname{ht}(I)+1\}$.
(b) *If* $\operatorname{ht}(I)>0$, $\operatorname{depth} R_A(I)=\min\{\dim A+1, \operatorname{depth} A/I+\operatorname{ht}(I)+2\}$.

Thus, by applying the computations made in Proposition 3.5 we have:

**Corollary 5.2.** *Let $(A,\mathfrak{m})$ be a local Cohen–Macaulay ring and $I\subseteq A$ an ideal with $\operatorname{ad}(I)=1$ and $r(I)\leq 1$. Assume that $I$ is generically a complete intersection. Then*

$$\operatorname{depth} G_A(I)=B(I)+l(I).$$

Let $I$ be an ideal and assume that there exists an element $a\in I$, $a\notin Z(I^r)$ with $I^{r+1}=aI^r$ for some integer $r\geq 1$. We will consider the following $G_A(I)$-modules:

$$U_r:=\bigoplus_{n\geq r} G_n=I^r/I^{r+1}\oplus\cdots\oplus I^n/I^{n+1}\cdots.$$

$$V_r:=U_r/atU_r\simeq I^r/aI^r$$

and

$$C_r:=A/I\oplus\cdots\oplus I^{r-1}/I^r.$$

Although some parts of the following lemmas have been shown in [15], we reprove them for completeness.

**Lemma 5.3** (See Goto et al. [15, Lemma 4.1]). *$at$ is a regular element of $U_r$.*

**Proof.** Let $x\in I^n$, $n\geq r$ with $ax\in I^{n+2}=aI^{n+1}$. Therefore there exists $y\in I^{n+1}$ such that $ax=ay$ and so $x-y\in(0:a)\cap I^r=0$. Hence $x\in I^{n+1}$. $\square$

**Lemma 5.4.** (a) $\operatorname{depth}_A I^r=\min\{d, \operatorname{depth} A/I^r+1\}$.
(b) $\operatorname{depth}_A I^r/aI^r=\min\{d-1, \operatorname{depth} A/I^r\}$.
(c) $\operatorname{depth}_G U_r=\min\{d, \operatorname{depth} A/I^r+1\}$.
(d) $\operatorname{depth}_G C_r=\min\{\operatorname{depth}_A I^n/I^{n+1},\ 0\leq n\leq r-1\}$.

**Proof.** Conditions (a) and (b) are obvious while (c) follows from Lemma 5.3. □

**Lemma 5.5** (See Goto et al. [15, Lemma 4.3]). *$a_i(U_r) \leq r-1$ $\forall i$. Furthermore, if $H^i_{\mathscr{M}}(U_r) \neq 0$ then $H^i_{\mathscr{M}}(U_r)_n \simeq H^i_{\mathscr{M}}(U_r)_{r-1}$ $\forall n \leq r-1$.*

**Proof.** From the exact sequence

$$0 \to U_r(-1) \xrightarrow{at} U_r \to V_r \to 0,$$

we have the long exact sequence of local cohomology

$$\cdots \to H^{i-1}_{\mathscr{M}}(V_r)_n \to H^i_{\mathscr{M}}(U_r)_{n-1} \to H^i_{\mathscr{M}}(U_r)_n \to H^i_{\mathscr{M}}(V_r)_n \to \cdots .$$

Since $H^i_{\mathscr{M}}(V_r)_n = 0$ for all $n \neq r$, we have isomorphisms $H^i_{\mathscr{M}}(U_r)_{n-1} \simeq H^i_{\mathscr{M}}(U_r)_n$ for all $n \neq r$. In particular $a_i(U_r) \leq r-1$. On the other hand, if $H^i_{\mathscr{M}}(U_r) \neq 0$ then $a_i(U_r) = r-1$ and $H^i_{\mathscr{M}}(U_r)_n \simeq H^i_{\mathscr{M}}(U_r)_{r-1}$ for all $n \leq r-1$. □

**Lemma 5.6** (See Goto et al. [15, Lemmas 4.4 and 4.5]). *$a_i(G) \leq r-1$ $\forall i$. Furthermore, if $\operatorname{depth} G = g < d$ then $a_g(G) < r-1$.*

**Proof.** From the exact sequence

$$0 \to U_r \to G \to C_r \to 0,$$

we obtain the long exact sequence of local cohomology

$$\cdots \to H^i_{\mathscr{M}}(U_r)_n \to H^i_{\mathscr{M}}(G)_n \to H^i_{\mathfrak{m}}(C_r)_n \to \cdots$$

for all $n$. Applying Lemma 5.5 and taking into account that $H^i_{\mathscr{M}}(C_r)_n = 0$ for all $n \geq r$ we obtain the first statement.

Assume now that $\operatorname{depth} G = g < d$. Then, $\operatorname{depth} R_A(I) = g+1$ by Lemma 2.12. Consider the exact sequences

$$0 \to R_+(1) \to R \to G \to 0, \tag{6}$$

$$0 \to R_+ \to R \to A \to 0. \tag{7}$$

From the long exact sequence of local cohomology associated to (6) we obtain isomorphisms $H^{g+1}_{\mathscr{M}}(R_+)_{n+1} \simeq H^{g+1}_{\mathscr{M}}(R)_n$ for all $n \geq r$ since $a_i(G) \leq r-1$. Similarly, from (7) we obtain isomorphisms $H^i_{\mathscr{M}}(R_+)_n \simeq H^i_{\mathscr{M}}(R)_n$ for all $n \neq 0$. Thus, for all $n \geq r$ we have isomorphisms $H^{g+1}_{\mathscr{M}}(R_+)_n \simeq H^{g+1}_{\mathscr{M}}(R_+)_{n+1}$.

If $a_g(G) = r-1$, (6) provides monomorphisms $H^g_{\mathscr{M}}(G)_{r-1} \hookrightarrow H^{g+1}_{\mathscr{M}}(R_+)_r$. Hence, $H^{g+1}_{\mathscr{M}}(R_+)_r \neq 0$ and so $H^{g+1}_{\mathscr{M}}(R_+)_n \neq 0$ for all $n \geq r$ which contradicts the fact that $a_{g+1}(R_+) < \infty$. □

**Proposition 5.7.** *If $a \notin Z_A(I^2)$ and $I^3 = aI^2$, then*

$$\begin{aligned}&\min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\}\\ &\quad \leq \operatorname{depth} G_A(I)\\ &\quad \leq \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + 1.\end{aligned}$$

*Furthermore, if* $\operatorname{depth} A/I \neq \operatorname{depth} A/I^2$ *and* $\operatorname{depth} A/I^2 < d$, *then*

$$\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2 + 1\}.$$

**Proof.** Throughout the proof we will consider the exact sequences

$$0 \to U_2 \to G \to C_2 \to 0, \tag{8}$$

$$0 \to I/I^2 \to A/I^2 \to A/I \to 0. \tag{9}$$

Moreover, we have by Lemma 5.4 that $\operatorname{depth}_G U_2 = \min\{d, \operatorname{depth} A/I^2 + 1\}$ and $\operatorname{depth}_G C_2 = \min\{\operatorname{depth} A/I, \operatorname{depth}_A I/I^2\}$.

Assume that $\operatorname{depth} A/I^2 = k < \operatorname{depth} A/I = t$. From sequences (9) and (8) we have that $\operatorname{depth} A/I^2 = \operatorname{depth} I/I^2$ and $\operatorname{depth} G \geq \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} = k$, respectively. On the other hand, from the long exact sequence of local cohomology associated to (8) we have monomorphisms $H^k_{\mathcal{M}}(G)_n \hookrightarrow H^k_{\mathcal{M}}(C_2)_n$ for all $n$. Hence $H^k_{\mathcal{M}}(G)_n = 0$ for $n \neq 0, 1$ and $H^k_{\mathcal{M}}(G)_0 \hookrightarrow H^k_{\mathfrak{m}}(A/I) = 0$. Therefore, $H^k_{\mathcal{M}}(G)_n = 0$ for all $n \neq 1$. If $\operatorname{depth} G = k$, then $H^k_{\mathcal{M}}(G)_1 \neq 0$ contradicting Lemma 5.6. Hence, $\operatorname{depth} G \geq k + 1$. We will show now that $H^{k+1}_{\mathcal{M}}(G) \neq 0$. By Lemma 5.5 we have $H^{k+1}_{\mathcal{M}}(U_2) \neq 0$. Moreover $H^k_{\mathcal{M}}(C_2)_0 \simeq H^k_{\mathfrak{m}}(A/I) = 0$. So, taking the component of degree 0 in the long exact sequence associated to (8) we have a monomorphism $H^{k+1}_{\mathcal{M}}(U_2)_0 \hookrightarrow H^{k+1}_{\mathcal{M}}(G)_0$. Hence, $H^{k+1}_{\mathcal{M}}(G)_0 \neq 0$.

Assume that $\operatorname{depth} A/I = t < \operatorname{depth} A/I^2 = k$. Applying the *depth-lemma* to (8) and (9) we obtain now that $\operatorname{depth} I/I^2 = \operatorname{depth} A/I + 1$ and $\operatorname{depth} G \geq \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} = t$. Moreover, if $k < d$ then $\operatorname{depth} G = \operatorname{depth} A/I$. Suppose that $k = d$. If $t < d - 1$, then from the long exact sequence of local cohomology associated to (8) we have an isomorphism $H^t_{\mathcal{M}}(G) \simeq H^t_{\mathcal{M}}(C_2) \neq 0$. Note that if $t = d - 1$, then $\operatorname{depth} G_A(I) \geq d - 1$.

Assume that $\operatorname{depth} A/I = \operatorname{depth} A/I^2 = t$. Again from (9) and (8) we get $\operatorname{depth} I/I^2 \geq \operatorname{depth} A/I$ and $\operatorname{depth} G \geq t$. In particular if $t = d$ then $G$ is Cohen–Macaulay. Assume that $t < d$. Taking the component of degree $n$ in the long exact sequence of graded local cohomology associated to (8) we have

$$0 \to H^t_{\mathcal{M}}(G)_n \to H^t_{\mathcal{M}}(C_2)_n \to H^{t+1}_{\mathcal{M}}(U_2)_n \to H^{t+1}_{\mathcal{M}}(G)_n.$$

So, $H^t_{\mathcal{M}}(G)_n = 0$ for all $n \neq 0, 1$. Moreover, by Lemma 5.6 we get that $\operatorname{depth} G = t$ if and only if $H^t_{\mathcal{M}}(G)_0 \neq 0$.

Finally, we will see that $\operatorname{depth} G \leq t+1$. We may assume that $t \leq d-2$ and $H^t_{\mathcal{M}}(G) = 0$. Then it suffices to see that $H^{t+1}_{\mathcal{M}}(G) \neq 0$. For all $n < 0$ we have monomorphisms $H^{t+1}_{\mathcal{M}}(U_2)_n \hookrightarrow H^{t+1}_{\mathcal{M}}(G)_n$. Thus, $H^{t+1}_{\mathcal{M}}(G)_n \neq 0$ for all $n < 0$ since $\operatorname{depth}_G U_2 = k+1$ and $H^{t+1}_{\mathcal{M}}(U_2)_n \neq 0$ for all $n \leq 1$ by Lemma 5.5. Therefore, we have that $t \leq \operatorname{depth} G \leq t+1$. Note that in any case the inequalities of the statement are satisfied. □

Now, we may formulate the main result of this section. In particular, for the Cohen–Macaulay property we recover [2, Theorem 3.2] and [14, Theorem 1.5].

**Theorem 5.8.** *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring and* $I \subseteq A$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$ *and* $r(I) \leq 2$. *Assume that* $I$ *is unmixed and* $r_h(I) \leq 1$, *where* $h = \operatorname{ht}(I)$.

*Then*

$$\begin{aligned}&\min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \operatorname{ht}(I)\\ &\quad\le \operatorname{depth} G_A(I)\\ &\quad\le \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \operatorname{ht}(I) + 1.\end{aligned}$$

*Furthermore, if* $\operatorname{depth} A/I \neq \operatorname{depth} A/I^2$ *and* $\operatorname{ht}(I) > 0$, *then*

$$\operatorname{depth} G_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2 + 1\} + \operatorname{ht}(I).$$

**Proof.** If $h = 0$ then $I^3 = aI^2$ for some $a \notin Z_A(I^2)$ and the statement follows from Proposition 5.7.

Assume that $h > 0$ and let $J$ be a minimal reduction of $I$ with $I^3 = JI^2$ and minimally generated by $a_1, \ldots, a_h$, $a$ as in Lemma 2.7. Put $A_h = A/(a_1, \ldots, a_h)$ and $I_h = I/(a_1, \ldots, a_h)$. Since the sequence $a_1^*, \ldots, a_h^*$ is regular in $G$ we have that $\operatorname{depth} G = \operatorname{depth} G_{A_h}(I_h) + h$, where $A_h$ is a Cohen–Macaulay ring and $I_h$ is an ideal with $\operatorname{ad}(I_h) = 1$, $\operatorname{ht}(I_h) = 0$, $r(I_h) \le 2$ and $r_0(I_h) \le 1$. Hence, by Proposition 5.7 we have that

$$\min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} \le \operatorname{depth} G_{A_h}(I_h) \le \min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} + 1.$$

Assume that $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$. Then, by Lemma 3.1(a)

$$\operatorname{depth} A/I^2 = \operatorname{depth} A_h/I_h^2 < \operatorname{depth} A/I.$$

Applying Proposition 5.7 we get thet that

$$\begin{aligned}\operatorname{depth} G &= \operatorname{depth} G(I_h) + h\\ &= \operatorname{depth} A_h/I_h^2 + 1 + h = \operatorname{depth} A/I^2 + h + 1.\end{aligned}$$

Suppose now that $\operatorname{depth} A/I < \operatorname{depth} A/I^2$. Then, from Lemma 3.1(b) and Proposition 5.7(b)
we get that $\operatorname{depth} A_h/I_h^2 = \operatorname{depth} A/I - 1$ and

$$\begin{aligned}\operatorname{depth} G &= \operatorname{depth} G(I_h) + h\\ &= \operatorname{depth} A_h/I_h^2 + 1 + h = \operatorname{depth} A/I - 1 + 1 + h = \operatorname{depth} A/I + h.\end{aligned}$$

Finally, if $\operatorname{depth} A/I = \operatorname{depth} A/I^2$ we get now by Proposition 5.7 that

$$\begin{aligned}\min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} &\le \operatorname{depth} G_{A_h}(I_h)\\ &\le \min\{\operatorname{depth} A/I, \operatorname{depth} A_h/I_h^2\} + 1\end{aligned}$$

and $\operatorname{depth} A_h/I_h^2 \ge \operatorname{depth} A/I - 1$ by Lemma 3.1. If $\operatorname{depth} A_h/I_h^2 = \operatorname{depth} A/I - 1$, then $\operatorname{depth} G_{A_h}(I_h) = \operatorname{depth} A/I$ and $\operatorname{depth} G = \operatorname{depth} A/I + h$. Otherwise $\operatorname{depth} A_h/I_h^2 \ge \operatorname{depth} A/I$ and so

$$\operatorname{depth} A/I \le \operatorname{depth} G_{A_h}(I_h) \le \operatorname{depth} A/I + 1.$$

Therefore,

$$\operatorname{depth} A/I + h \le \operatorname{depth} G \le \operatorname{depth} A/I + h + 1. \qquad \square$$

**Example 5.9.** Let $A$ and $\mathfrak{p}$ be as in Example 2.9. Then, one may verify by using CoCoA [7] that $\operatorname{depth} A/\mathfrak{p} = 1$. Since $\dim A/\mathfrak{p} = 2$ we have, by Proposition 3.5, that $\operatorname{depth} A/\mathfrak{p}^n = 1$ for all $n \geq 1$. So, by applying the above Theorem 5.8 to the ideal $J = \mathfrak{p}^2$ we obtain $3 \leq \operatorname{depth} G_A(J) \leq 4$. In fact, $\operatorname{depth} G_A(J) = 4$ since $G_A(\mathfrak{p})$ is Cohen–Macaulay [19, Proposition-Example 4.1] and so $G_A(\mathfrak{p}^2)$ too.

**Example 5.10.** Let $A = k[a,b,c,d,e]$ and $I = (f_1, f_2, f_3)$ be as in Example 2.10. Let $B = A/f_1^2$ and $J = I/f_1^2$. Then, $\operatorname{depth} B/J^2 = \operatorname{depth} A/I^2 = \operatorname{depth} A/I = 0$ by Proposition 3.5 and so $1 \leq \operatorname{depth} G_B(J) \leq 2$ by Theorem 5.8. In particular, $G_B(J)$ is not Cohen–Macaulay.

Again, by applying the computation of the Burch number made in Proposition 3.6 we get

**Corollary 5.11.** *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring and* $I \subseteq A$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$, $r(I) \leq 2$ *and* $r_h(I) \leq 1$, *with* $h = \operatorname{ht}(I)$. *Assume that* $I$ *is unmixed. Then*

$$B(I) + l(I) - 1 \leq \operatorname{depth} G_A(I) \leq B(I) + l(I) + 1.$$

*Furthermore, if* $\operatorname{depth} A/I \neq \operatorname{depth} A/I^2$ *and* $\operatorname{ht}(I) > 0$, *then*

$$\operatorname{depth} G_A(I) = B(I) + l(I).$$

As for the Rees algebra we obtain by Lemma 2.12 the following values for its depth:

**Theorem 5.12.** *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring and* $I \subseteq A$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$, $r(I) \leq 2$ *and* $r_h(I) \leq 1$, *with* $h = \operatorname{ht}(I)$. *Assume that* $I$ *is unmixed and* $\operatorname{ht}(I) \geq 2$. *Then*

$$\begin{aligned} &\min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \operatorname{ht}(I) + 1 \\ &\quad \leq \operatorname{depth} R_A(I) \\ &\quad \leq \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2\} + \operatorname{ht}(I) + 2. \end{aligned}$$

*Furthermore, if* $\operatorname{depth} A/I \neq \operatorname{depth} A/I^2$, *then*

$$\operatorname{depth} R_A(I) = \min\{\operatorname{depth} A/I, \operatorname{depth} A/I^2 + 1\} + \operatorname{ht}(I) + 1.$$

## 6. Powers of ideals

In this section we will compare the depths of the graded rings associated to an ideal $I$ with the depths of the graded rings associated to its powers $I^n$ for the class of ideals treated throughout the above sections. We will use the following lemma proved by Ribbe [25].

**Lemma 6.1.** *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an ideal. Let $n \in \mathbb{N}$ be a positive integer and $g := \operatorname{depth} G_A(I)$.*
(a) *If $H^g_{\mathcal{M}}(G_A(I))_j \neq 0$, $\forall j \ll 0$, then $\operatorname{depth} G_A(I^n) = \operatorname{depth} G_A(I)$.*
(b) *If $\operatorname{ht}(I) > 0$ and $H^{g+1}_{\mathcal{M}}(R_A(I))_j \neq 0$, $\forall j \ll 0$, then $\operatorname{depth} R_A(I^n) = \operatorname{depth} R_A(I)$.*

**Proof.** Condition (b) is clear since $R(I^n) = R(I)^{(n)}$ for all $n$. For (a) apply [25, Lemma 5.3.1]. $\square$

For equimultiple ideals we obtain the following results.

**Lemma 6.2.** *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal with $r(I) \leq 1$. Put $g := \operatorname{depth} G_A(I)$.*
(a) *If $\operatorname{ht}(I) \geq 1$, then $H^g_{\mathcal{M}}(G_A(I))_j \neq 0$ for all $j \leq -\operatorname{ht}(I)$.*
(b) *If $\operatorname{ht}(I) \geq 2$, then $H^{g+1}_{\mathcal{M}}(R_A(I))_j \neq 0$ for all $j \leq -\operatorname{ht}(I) + 1$.*

**Proof.** We prove (a) by induction on $\operatorname{ht}(I)$. We first consider the case $h = 1$. Then, by Theorem 4.1 one has that $g = \operatorname{depth} A/I + 1$. On the other hand, $I^2 = aI$ for some $a \notin Z(A)$. Using the notations of Section 5 we have the long exact sequence of local cohomology

$$\cdots \to H^{g-1}_{\mathcal{M}}(A/I)_j \to H^g_{\mathcal{M}}(U_1)_j \to H^g_{\mathcal{M}}(G)_j \to \cdots .$$

In particular, for $j \leq -1$ we have injections $H^g_{\mathcal{M}}(U_1)_j \hookrightarrow H^g_{\mathcal{M}}(G)_j$. Moreover, by Lemmas 5.4 and 5.5 we have, respectively, that $\operatorname{depth} U_1 = g$ and $H^g_{\mathcal{M}}(U_1)_j \neq 0$ for all $j \leq 0$. Hence, $H^g_{\mathcal{M}}(G)_j \neq 0$ for all $j \leq -h$ as we wanted.

Now assume that $h > 1$. Let $J = (a_1, \ldots, a_h)$ be a minimal reduction of $I$ with $I^2 = JI$. Put $A_1 = A/(a_1)$ and $I_1 = I/(a_1)$. The ideal $I_1$ is equimultiple with $r(I_1) \leq 1$ and $\operatorname{ht}(I_1) = h - 1$. Moreover, since $a_1^*$ is regular in $G$, we have the exact sequence

$$0 \to G(-1) \xrightarrow{a_1^*} G \to G(I_1) \to 0$$

that provides in cohomology injections

$$0 \to H^{g-1}_{\mathcal{M}}(G(I_1))_j \to H^g_{\mathcal{M}}(G)_{j-1}.$$

By induction $H^{g-1}_{\mathcal{M}}(G(I_1))_j \neq 0$ for all $j \leq -\operatorname{ht}(I_1) = 1 - h$. Thus, $H^g_{\mathcal{M}}(G)_j \neq 0$ for all $j \leq -h$ and (a) is shown.

For (b) assume that $\operatorname{ht}(I) \geq 2$. Then, by Proposition 4.1(b) we have that $\operatorname{depth} R = g + 1$. Furthermore, from the exact sequences (6) and (7) in Lemma 5.6 we obtain in cohomology isomorphisms $H^{g+1}_{\mathcal{M}}(R_+)_j \simeq H^{g+1}_{\mathcal{M}}(R)_j$ for all $j \neq 0$ and monomorphisms $H^g_{\mathcal{M}}(G)_j \hookrightarrow H^{g+1}_{\mathcal{M}}(R_+)_{j+1}$ for all $j$. From (a) we have that $H^g_{\mathcal{M}}(G)_j \neq 0$ for all $j \leq -h$ and so $H^{g+1}_{\mathcal{M}}(R_+)_j \neq 0$ for all $j \leq -h+1$. Hence, we may conclude that $H^{g+1}_{\mathcal{M}}(R)_j \neq 0$ for all $j \leq -h+1$. $\square$

**Proposition 6.3.** *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay local ring, $I$ an equimultiple ideal with $r(I)\leq 1$ and $n\in\mathbb{N}$ a positive integer.*
(a) *If* $\mathrm{ht}(I)\geq 1$, *then* $\operatorname{depth} G_A(I^n)=\operatorname{depth} G_A(I)$.
(b) *If* $\mathrm{ht}(I)\geq 2$, *then* $\operatorname{depth} R_A(I^n)=\operatorname{depth} R_A(I)$.

**Proof.** Apply Lemmas 6.1 and 6.2. □

**Remark 6.4.** The above result is not necessarily true for ideals with $\mathrm{ht}(I)=0$. In this case we have that $\operatorname{depth} G_A(I)=\operatorname{depth} A/I$ whereas $\operatorname{depth} G_A(I^n)=\operatorname{depth} A$ for $n\geq 2$.

**Lemma 6.5.** *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal. Assume that $r(I)\leq 2$ and either $I$ is integrally closed or $I$ is unmixed with $r_h(I)\leq 1$. Suppose that* $\operatorname{depth} A/I^2<\operatorname{depth} A/I$ *and put* $g:=\operatorname{depth} G_A(I)$.
(a) *If* $\mathrm{ht}(I)\geq 1$, *then* $H^g_{\mathscr{M}}(G_A(I))_j\neq 0\quad \forall j\leq -\mathrm{ht}(I)$.
(b) *If* $\mathrm{ht}(I)\geq 3$, *then* $H^{g+1}_{\mathscr{M}}(R(I))_j\neq 0\quad \forall j\leq -\mathrm{ht}(I)+1$.

**Proof.** We proceed by induction on $h$. Assume that $h=1$. Then $I^3=(a)I^2$ for some $a\notin Z(A)$ and, by Theorem 4.5 (or Proposition 5.7), we have that $g=\operatorname{depth} A/I^2+1$. Furthermore, from the long exact sequence of local cohomology associated to (8) we obtain for all $j$

$$\cdots\to H^{g-1}_{\mathscr{M}}(C_2)_j\to H^g_{\mathscr{M}}(U_2)_j\to H^g_{\mathscr{M}}(G)_j\to\cdots .$$

In particular, for all $j\leq -1$ one has monomorphisms $H^g_{\mathscr{M}}(U_2)_j\hookrightarrow H^g_{\mathscr{M}}(G)_j$. Moreover, $\operatorname{depth} U_2=g$ and $H^g_{\mathscr{M}}(U_2)_j\neq 0$ for all $j\leq 1$ by Lemma 5.5. Thus, $H^g_{\mathscr{M}}(G)\neq 0$ for all $j\leq -1$.

Assume that $h>1$ and let $J=(a_1,\ldots,a_h)$ be a minimal reduction of $I$ with $I^3=JI^2$. Then, by Lemma 2.4, the sequence $a_1^*,\ldots,a_h^*$ is regular in $G$. Put $A_{h-1}=A/(a_1,\ldots,a_{h-1})$ and $I_{h-1}=I/(a_1,\ldots,a_{h-1})$. Then $I_{h-1}$ is an equimultiple ideal in the Cohen–Macaulay ring $A_{h-1}$ with $\mathrm{ht}(I_{h-1})=1$, $r(I_{h-1})\leq 2$ and $\operatorname{depth} A_{h-1}/I^2_{h-1}=\operatorname{depth} A/I^2<\operatorname{depth} A/I=\operatorname{depth} A_{h-1}/I_{h-1}$. Therefore, $H^{g-h+1}_{\mathscr{M}}(G(I_{h-1}))_j\neq 0$ for all $j\leq -1$.

Considering, for all $i=1,\ldots,h-1$ $(I_0:=I)$, the exact sequences

$$0\to G(I_{i-1})(-1)\xrightarrow{a_i^*}G(I_{i-1})\to G(I_i)\to 0,$$

we obtain in cohomology injections

$$0\to H^{g-i}_{\mathscr{M}}(G(I_i))_j\to H^{g-i+1}_{\mathscr{M}}(G(I_{i-1}))_{j-1}.$$

Then, by descendent induction on $i$, we have that $H^g_{\mathscr{M}}(G)_j\neq 0$ for all $j\leq -h$ and (a) is proved.

To show (b) we may proceed as in Lemma 6.2 taking into account that in this case $\operatorname{depth} R=\operatorname{depth} G+1$ if $\mathrm{ht}(I)\geq 3$. □

**Proposition 6.6.** *Let $(A,\mathfrak{m})$ be a Cohen–Macaulay local ring and $I$ an equimultiple ideal. Assume that $r(I)\leq 2$ and either $I$ is integrally closed or $I$ is unmixed with*

$r_h(I) \leq 1$. *Suppose that* $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$ *and let* $n \in \mathbb{N}$ *be a positive integer.*

(a) *If* $\operatorname{ht}(I) \geq 1$, *then* $\operatorname{depth} G_A(I^n) = \operatorname{depth} G_A(I)$.
(b) *If* $\operatorname{ht}(I) \geq 3$, *then* $\operatorname{depth} R_A(I^n) = \operatorname{depth} R_A(I)$.

**Proof.** The result follows from Lemmas 6.1 and 6.5. □

**Example 6.7.** Consider again the ideal $I = (xy, t_1, \ldots, t_n)$ given in Example 2.8. In this case we have that $\operatorname{depth} G_A(I^m) = n$ and $\operatorname{depth} R_A(I^m) = n + 1$ for all $m \geq 1$.

The following results for analytic deviation one ideals may be obtained in a similar form. For their proofs we can adapt the arguments used in the equimultiple case and so we omit them. Note that in the case of a generically complete intersection ideal of analytic deviation 1 and reduction number less or equal than 1 we recover [32, Theorem 4.3].

**Lemma 6.8.** *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring and* $I$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$ *and* $r(I) \leq 2$. *Assume that* $I$ *is unmixed and* $r_h(I) \leq 1$. *Suppose that* $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$ *and put* $g := \operatorname{depth} G_A(I)$. *Then*
(a) $H^g_{\mathscr{M}}(G_A(I))_j \neq 0 \quad \forall j \leq -\operatorname{ht}(I) - 1$.
(b) *If* $\operatorname{ht}(I) \geq 2$, *then* $H^{g+1}_{\mathscr{M}}(R(I))_j \neq 0 \quad \forall j \leq -\operatorname{ht}(I)$.

**Proposition 6.9.** *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring and* $I$ *an ideal of* $A$ *with* $\operatorname{ad}(I) = 1$ *and* $r(I) \leq 2$. *Assume that* $I$ *is unmixed and* $r_h(I) \leq 1$. *Suppose that* $\operatorname{depth} A/I^2 < \operatorname{depth} A/I$ *and let* $n \in \mathbb{N}$ *be a positive integer. Then*
(a) $\operatorname{depth} G_A(I^n) = \operatorname{depth} G_A(I)$.
(b) *If* $\operatorname{ht}(I) \geq 2$, *then* $\operatorname{depth} R_A(I^n) = \operatorname{depth} R_A(I)$.

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